Exam VI
Section I
Part A — No Calculators

1. A p. 121

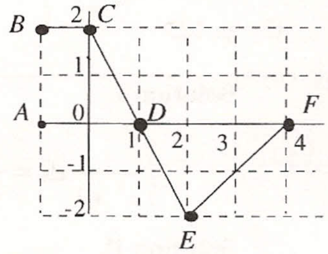
Using the product rule we obtain the first and second derivatives of $y = xe^x$.

$$y' = e^x + xe^x$$
$$y'' = e^x + e^x + xe^x$$
$$= 2e^x + 2e^x$$
$$= e^x(2 + x)$$

Thus y'' changes sign at $x = -2$.

2. C p. 121

Since $H(4) = \int_1^4 f(t)\, dt$, then



$H(4) =$ area trapezoid ABCD – area triangle DEF

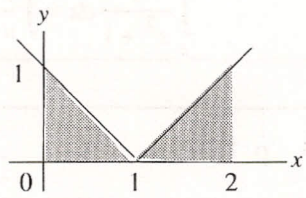
$$= \frac{1}{2}(2)(1+2) - \frac{1}{2}(3)(2) = 3 - 3$$
$$= 0$$

3. B p. 122

Solution I.

$$\int_0^2 |x-1|\, dx = \text{area of 2 equal triangles}$$

$$= 2\left(\frac{1}{2}\right)(1)(1) = 1.$$



Solution II.

$$\int_0^2 |x-1|\, dx = \int_0^1 -x+1\, dx + \int_1^2 x-1\, dx = -\frac{x^2}{2} + x\Big|_0^1 + \frac{x^2}{2} - x\Big|_1^2 = \frac{1}{2} + \frac{1}{2} = 1$$

4. E p. 122

(A), (B), and (D) are part of the definition of continuity and hence true. (C) is a statement equivalent to (A). (E) is the only one that could be false, for example, consider $y = |x|$ at $x = 0$.

5. C p. 122

$$\int_0^x 2\sec^2\left(2t+\frac{\pi}{4}\right)dt = \tan\left(2t+\frac{\pi}{4}\right)\Big|_0^x = \tan\left(2x+\frac{\pi}{4}\right) - \tan\frac{\pi}{4} = \tan\left(2x+\frac{\pi}{4}\right) - 1.$$

6. A p. 123

Using implicit differentiation on $xy + x^2 = 6$ we obtain

$$y + xy' + 2x = 0$$
$$xy' = -2x - y$$
$$y' = \frac{-2x-y}{x}$$

At $x = 1$

$-y + 1 = 6$ and $y = -5$

Thus $y' = \dfrac{2+5}{-1} = -7$

7. C p. 123

Solution I.

$$\int_2^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_2^3 \frac{1}{x^2+1}(2x\,dx) = \frac{1}{2}\ln(x^2+1)\Big|_2^3 = \frac{1}{2}[\ln 10 - \ln 5] = \frac{1}{2}\ln 2$$

Solution II.

Let $u = x^2 + 1$, then $du = 2x\,dx$. Then

$$\int_2^3 \frac{x}{x^2+1}\,dx = \int_5^{10} \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln u\Big|_5^{10} = \frac{1}{2}(\ln 10 - \ln 5) = \frac{1}{2}\ln 2.$$

8. B p. 123

$f'(x) = e^{\sin x}$

I. $f''(x) = e^{\sin x}\cos x$ $\quad f''(0) = 1\cdot 1 = 1$ TRUE

II. Slope of $y = x + 1$ is 1, and the line goes through (0, 1) TRUE

III. $h(x) = f(x^3 - 1)$ $\quad h'(x) = f'(x^3-1)\cdot(3x^2) = 3x^2 e^{\sin(x^3-1)}$

Since $h'(x) \geq 0$ for all x, then the graph of $h(x)$ is increasing. TRUE

9. D p. 124

The total flow is $\int_0^6 f(t)\,dt$ which is equal to the area under the curve. Each square under the curve represents 10 gallons. There are 11 full squares and 2.5 part squares. Hence, the approximate area is $13.5 \times 10 = 135$ gallons.

10. B p. 124

The instantaneous rate of change of $f(x) = e^{2x} - 3\sin x$ is

$$f'(x) = 2e^{2x} - 3\cos x.$$

Thus, $f'(0) = 2e^{0} - 3\cos 0 = 2 - 3 = -1$.

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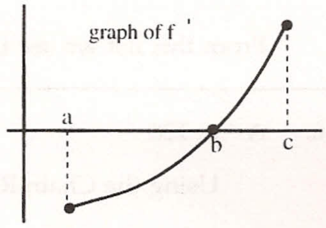
11. E p. 124

f' is negative on (a, b) so f is decreasing there and (A) is false. f' increases on (a, c), hence, f'' is positive and the graph of f is concave up. As a result, (B), (C), and (D) are also false.
(E) is true because f is an antiderivative of f'.



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12. E p. 125

Using the Fundamental Theorem and the Chain Rule on

$F(x) = \int_0^{x^2} \frac{1}{2+t^3}\,dt$, we obtain $F'(x) = \frac{1}{2+(x^2)^3} \cdot 2x$.

Then, $F'(-1) = \frac{1}{2+1} \cdot (-2) = -\frac{2}{3}$.

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13. D p. 125

f is not differentiable at $x = 0$ because $\lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x)$.

f is not differentiable at $x = 3$ because f is not continuous there.

Hence the answer is (D).

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14. D p. 125

The velocity $v(t) = x'(t) = \frac{(t^2+4)(1) - t(2t)}{(t^2+4)^2} = \frac{4-t^2}{(t^2+4)^2} = \frac{(2+t)(2-t)}{(t^2+4)^2}$.

For $t \geq 0$, $v(t) = 0$ when $t = 2$.

15. D p. 126

To find the maximum value of $f(x) = 2x^3 + 3x^2 - 12x + 4$ on the closed interval [0,2], we evaluate the function f at the critical numbers and endpoints and take the largest resulting value.

$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$.

Thus in the closed interval [0,2], the only critical number is $x = 1$. The values of the function at the one critical number and at the endpoints are

critical value: $f(1) = -3$
endpoints: $f(0) = 4$ and $f(2) = 8$

From this list we see that the maximum value is 8.

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16. A p. 126

Using the Chain Rule twice on $f(x) = \ln(\cos 2x)$, we obtain

$$f'(x) = \frac{1}{\cos 2x} \cdot (-\sin 2x) \cdot (2).$$

$$= -2\frac{\sin 2x}{\cos 2x} = -2\tan 2x.$$

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17. A p. 126

The slopes of the segments in the slope field vary as x changes. Thus the differential equation is not of the form $\frac{dy}{dx} = g(y)$. That eliminates B, C and E of the five choices. If the differential equation were $\frac{dy}{dx} = x^2 + y^2$, then slopes would always be at least 0. They aren't in Quadrant III. That eliminates choice (D).

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18. E p. 127

$$y = \sqrt{x+3} = (x+3)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x+3)^{-\frac{1}{2}}$$

At (1,2), $y' = \frac{1}{2}\frac{1}{\sqrt{1+3}} = \frac{1}{4}$ and the equation of the line is $y - 2 = \frac{1}{4}(x-1)$.

When $x = 0$, then $y - 2 = \frac{1}{4}(0-1)$ and $y = \frac{7}{4}$.

19. B p. 127

Solution I.

Using the Product Rule twice on $f(x) = (x-1)(x+2)^2$ gives

$f'(x) = (x+2)^2 + (x-1)\cdot 2(x+2)$

$f''(x) = 2(x+2) + 2(x+2) + 2(x-1)$

$= 6x + 6$

Thus, $f''(x) = 0$ when $x = -1$.

Solution II.

Multiplying out $f(x) = (x-1)(x+2)^2$ gives

$f(x) = (x-1)(x+2)^2$

$= (x-1)(x^2 + 4x + 4) = x^3 + 3x^2 - 4$

$f'(x) = 3x^2 + 6x$

$f''(x) = 6x + 6$, thus $f''(x) = 0$ when $x = -1$.

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20. C p. 127

$$\int_0^b (4bx - x^2)\,dx = \frac{4bx^2}{2} - \frac{2x^3}{3}\Bigg|_0^b = 2b^3 - \frac{2b^3}{3} = \frac{4b^3}{3}$$

Thus $\frac{4b^3}{3} = 36 \Rightarrow b^3 = 27$ and $b = 3$.

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21. C p. 128

Separating variables for $\frac{dy}{dx} = -10y$, gives: $\frac{1}{y}\,dy = -10\,dx$.

Integrating gives $\ln|y| = -10x + c$.

Thus, $|y| = e^{-10x+c} = e^{-10x} \cdot e^{c}$ and $y = De^{-10x}$ where $D = \pm e^{c}$.

Substituting $y = 50$ and $x = 0$, gives: $50 = De^{0}$ and D = 50.

Thus, $y = 50e^{-10x}$.

22. C p. 128

$f(x) = x^3 - 5x^2 + 3x$

$f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3)$ and $f'(x) < 0$ when $\frac{1}{3} < x < 3$.

$f''(x) = 6x - 10$ and $f''(x) < 0$ when $x < \frac{5}{3}$.

Thus $\frac{1}{3} < x < \frac{5}{3}$ satisfies both.

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23. C p. 128

I is false as $f'(x) < 0$ on (1, 3), hence f is decreasing on part of (2, 4).

II is false as $f'(x)$ goes from plus to minus at $x = 2$, hence f has a relative maximum there.

III is true as f' has a horizontal tangent there which means that $f'(x) = 0$ and since the slope of f' changes sign there is an inflection point at $x = 1$.
Thus the answer is C.

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24. B p. 129

$f(x) = \arctan(2x - x^2)$

$f'(x) = \dfrac{2-2x}{1+(2x-x^2)^2}$

$f'(x) = 0$ when $x = 1$.

$1 + (2x - x^2)^2 > 0$, so f' exists for all $x \in \Re$.

Thus $x = 1$ is the only critical value.

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25. C p. 129

I is true because $\lim_{x\to 1}|x-1| = 0$ and $f(1) = 0$, so f is continuous at $x = 1$.

II is true because $y = e^x$ is continuous for all real x and $y = x - 1$ is continuous for all x, hence the composite is continuous for all x.

III is false because at $x = 1$, $\ln(e^0 - 1) = \ln(1-1) = \ln 0$ which is undefined.

Thus the answer is (C).

26. D p. 129

The number of motels is

$$\int_0^5 m(x)\,dx = \int_0^5 11 - e^{0.2x}\,dx = 11x - \frac{1}{0.2}e^{0.2x}\Big|_0^5 = 11x - 5e^{0.2x}\Big|_0^5$$

$$= (55 - 5e^1) - (0 - 5e^0)$$

$$= 55 - 5e + 5$$

$$= 60 - 5e$$

Since $e \approx 2.71$ and $5e \approx 13.55$, the number of motels is approximately $60 - 13.55 \approx 46$.

(*N.B.* Use $e \approx 3 \Rightarrow 5e \approx 15 \Rightarrow 60 - 5e \approx 45$ and (D) is still the best approximation.)

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27. C p. 130

The relationship between x, y, and z is Pythagorean: $z^2 = x^2 + y^2$.
The derivative with respect to time, t, gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \quad (*)$$

When $x = 3$ and $y = 4$, then $z = 5$.

Substituting $3\frac{dx}{dt}$ for $\frac{dy}{dt}$ and 2 for $\frac{dz}{dt}$, in (*) we obtain

$$(5)(2) = (3)\frac{dx}{dt} + 4\left(3\frac{dx}{dt}\right)$$

$$10 = 15\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2}{3}.$$

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28. E p. 130

$$f(x) = \sin(2x) + \ln(x+1)$$

$$f'(x) = \cos(2x)\cdot 2 + \frac{1}{x+1}$$

$$f'(0) = \cos(0)\cdot 2 + \frac{1}{0+1} = 2 + 1 = 3.$$

Exam VI
Section I
Part B — Calculators Permitted

1. E p. 131

(A) is true because in the neighborhood of $x = a$, $f(a)$ is the smallest value.
(B) and (C) and (D) are true from reading the graph.
(E) is false because f' does not exist at $x = a$. (Use the contrapositive of the theorem: If $f'(a)$ exists, then f is continuous at $x = a$; that is, if f is not continuous at $x = a$, then $f'(a)$ does not exist.)

2. C p. 131

A horizontal tangent occurs when $f'(x) = 0$.

$$f(x) = e^{3x} + 6x^2 + 1$$

$$f'(x) = 3e^{3x} + 12x$$

Thus $f'(x) = 0$ when $x = -0.156$.

3. C p. 132

Differentiating $PV = c$ with respect to t gives $\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0$.

At $P = 100$ and $V = 20$ and $\frac{dV}{dt} = -10$, we have $\frac{dP}{dt} \cdot 20 + 100(-10) = 0$

$$\frac{dP}{dt} = \frac{1000}{20} = 50 \frac{\text{lb/in}^2}{\text{sec}}$$

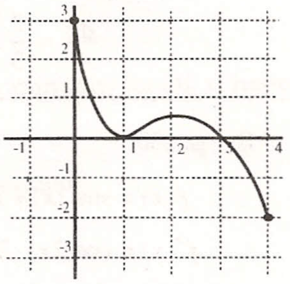
4. C p. 132

I is false because g'' changes sign only at $x = 3$.

II is true because $g''(x) < 0$ on (3, 4).

III is true because $g''(x) \geq 0$ on (0, 2), thus $g'(x)$ is non-decreasing there. Since $g'(0) = 0$ and g' does not decrease, then $g'(x) > 0$ at $x = 2$ and g is increasing there. Thus, the answer is C.

Graph of g''



5. E p. 132

To find the maximum velocity on the closed interval [0, 3], we evaluate the velocity function at the critical numbers and endpoints and take the largest resulting value.

Using the Fundamental Theorem to find the derivative of $s(t) = \int_0^t (x^3 - 2x^2 + x)\,dx$ gives

$v(t) = s'(t) = t^3 - 2t^2 + t$

Solution I.

$v'(t) = 3t^2 - 4t + 1 = (3t - 1)(t - 1)$

Thus on the closed interval [0, 3], the critical number are $t = 1$ and $t = \frac{1}{3}$.
The values of the velocity function at the critical numbers and at the endpoints are

critical value: $v\left(\frac{1}{3}\right) = \frac{4}{27}$ and $v(1) = 0$

endpoints: $v(0) = 0$ and $v(3) = 12$

From this list we see that the maximum velocity is 12 m/sec

Solution II.

Using a calculator to graph $v(t)$ in a $[0,3]$ x $[0,2]$ viewing rectangle we see that $v(t)$ has a relative maximum between $x = 0$ and $x = 1$ and a relative minimum at $x = 1$, but increases steeply to the absolute maximum at the right-hand end point (3, 12).

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6. D p. 133

Solution I. The rate of change $= \frac{dy}{dx} = \sqrt{2x+1}$.

On [0, 4], the average rate of change $= \frac{1}{4-0}\int_0^4 \sqrt{2x+1}\,dx$

$$= \frac{1}{4}\int_0^4 (2x+1)^{1/2}\,dx = \frac{1}{4}\cdot\left[\frac{1}{2}(2x+1)^{3/2}\cdot\frac{2}{3}\right]$$

$$= \frac{1}{12}\left[9^{3/2} - 1\right] = \frac{1}{12}[27 - 1] = \frac{13}{6}.$$

Solution II. Average rate of change $= \frac{1}{4}$ fnInt$\left(\sqrt{2x+1}, x, 0, 4\right) = 2.1666$

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7. A p. 133

Since $f(x)$ is an antiderivative of $f'(x)$, then $\int_0^2 f'(x)\,dx = f(2) - f(0) = 0$.

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8. B p. 134

$$\lim_{x\to k} \frac{x^2 - k^2}{x^2 - kx} = \lim_{x\to k} \frac{(x+k)(x-k)}{x(x-k)} = \lim_{x\to k} \frac{(x+k)}{x} = 2$$

9. B p. 134

Let w = the weight of the duck, then $\frac{dw}{dt} = kw$.

Separating the variables gives $\frac{1}{w}dw = k\,dt$.

Integrating yields $\ln w = kt + c$ When $t = 0$ and $w = 2$, then $C = 2$,

$w = e^{kt+c} = Ce^{kt}$. when $t = 4$ and $w = 3.5$, then $3.5 = 2e^{4k}$.

Thus $4k = \ln 3.5$ and $k = 0.140$. At $t = 6$, $w = 2e^{0.14(6)} = 4.63$.

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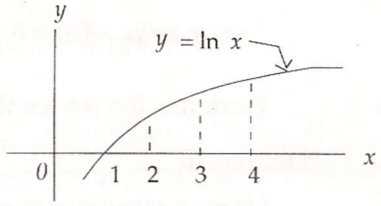
10. D p. 135

$\Delta x = \frac{4-1}{3} = 1$ $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4.$



$T_3 = \frac{1}{2}(\Delta x)[y_0 + 2y_1 + 2y_2 + y_3]$

$= \frac{1}{2}(1)[0 + 2\ln 2 + 2\ln 3 + \ln 4]$

$= 2.4849$

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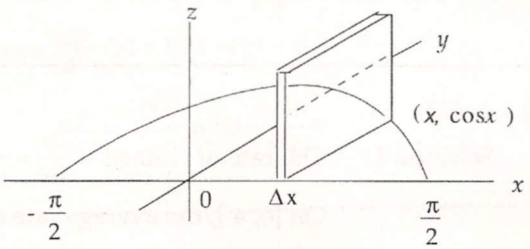
11. C p. 135

$V = 2\int_0^{\pi/2} (\cos x)^2\, dx = 1.5708$



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12. E p. 136

Consider points $P = (a-h, f(a-h))$, $Q = (a, f(a))$, and $R = (a+h, f(a+h))$.
The definition of the derivative of f at $x = a$ is the limit of the slope of the secant.

For $\overline{PQ}$ $f'(a) = \lim\limits_{h\to 0} \frac{f(a)-f(a-h)}{a-(a-h)} = \lim\limits_{h\to 0} \frac{f(a)-f(a-h)}{h}$. II

For $\overline{QR}$ $f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{a+h-a} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$. I

For $\overline{PR}$ $f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a-h)}{(a+h)-(a-h)} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$. III

Since f is differentiable, all the limits exist and all three are true.

13. A p. 136

$$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Using fnInt or graphing and using ∫f(x) dx for $L(x) = e^{-0.1x} + \frac{1}{x^2}$ on $a = 15$ to $b = 25$

gives $L_{ave} = \frac{1}{25-15}\int_{15}^{25} e^{-0.1x} + \frac{1}{x^2}\,dx = \frac{1}{10}(1.437) \approx 0.144.$

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14. D p. 137

The graph of $y = f'(x) = e^x\sin(x^2) - 1$ has three zeros on [0, 3]. $x = 0.715$ is a relative minimum, $x = 1.721$ is a relative maximum, and $x = 2.523$ is a relative minimum. Thus I and II are true.

The graph of f' has three horizontal tangents on [0, 3], so f has 3 inflection points and III is false.

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15. B p. 137

$$y_{ave} = \frac{1}{b-1}\int_1^b x^2\,dx = \frac{1}{b-1}\left.\frac{x^3}{3}\right|_1^b = \frac{1}{3(b-1)}(b^3-1).$$

Solution I. Using the calculator to find the zeros of

$\frac{1}{3(b-1)}(b^3-1) = \frac{13}{3}$, we obtain $b = 3$.

Solution II. $\frac{1}{3(b-1)}(b^3-1) = \frac{(b-1)(b^2+b+1)}{3(b-1)} = \frac{b^2+b+1}{3}$.

Solving $\frac{b^2+b+1}{3} = \frac{13}{3}$. gives $b^2+b+1 = 13$,

then $b^2+b+12 = 0$, and $(b+4)(b-3) = 0$, so $b = -4$ or $b = 3$.

The only solution on $[1, b]$ is $b = 3$.

16. C p. 138

$$f(x) = \frac{2x}{x^2 + 1}$$

Substituting $u = x^2 + 1$ and $du = 2x\,dx$ with $x = 0 \Rightarrow u = 1$ and

$x = 3 \Rightarrow u = 10$ into $\int_0^3 f(x)\,dx$ gives $\int_1^{10} \frac{1}{u}\,du = \ln|u| \Big|_1^{10} = \ln 10$.

Thus I is true.

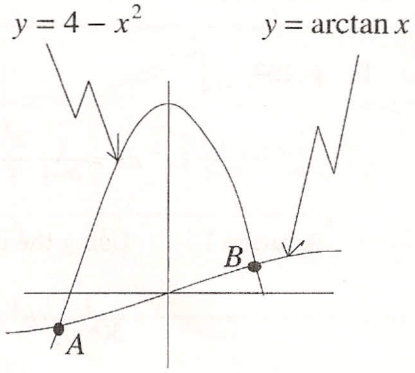
$$f'(x) = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2} = \frac{2(1-x)(1+x)}{(x^2+1)^2}.$$

From f' we see the critical values are ±1. At $x = 1$, $f'(x)$ goes from positive to negative, so (1, 1) is a relative maximum and II is true.

$f'(2) = \frac{2(1-2)(1+2)}{(2^2+1)^2} = -\frac{6}{25}$, thus III is false.

17. B p. 138

Using the graphing calculator, we graph $y = \arctan x$ and $y = 4 - x^2$ on the same axes. Next determine the intersection points $A = (-2.270, -1.156)$ and $B = (1.719, 1.044)$.



The area under $y = 4 - x^2$ and above $y = \arctan x$ is given by

$$\int_{-2.270}^{1.719} [(4 - x^2) - \arctan x]\,dx = 10.972.$$

Exam VI
Section II
Part A — Calculators Permitted

1. p. 140

(a)

$$\text{Area} = \int_1^8 \frac{8}{\sqrt[3]{x}}\,dx = \int_1^8 8x^{-1/3}\,dx = 8x^{2/3}\cdot\frac{3}{2}$$

$$= 12(8)^{2/3} - 12(1)^{2/3} = 48 - 12 = 36.$$

$$3:\begin{cases}1:\text{limits}\\ 1:\text{integrand}\\ 1:\text{answer}\end{cases}$$

(b) The area from $x = 1$ to $x = k$ is $\int_1^k 8x^{-1/3}\,dx = 12k^{2/3} - 12$.

Solving $12k^{2/3} - 12 = \frac{5}{12}\cdot 36 = 15$ gives $k^{2/3} = \frac{15+12}{12} = \frac{9}{4}$

$$k = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \frac{27}{8} = 3.375.$$

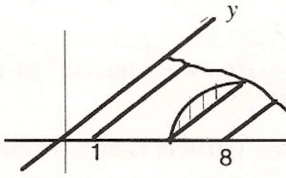
$$3:\begin{cases}2:\text{definite integral}\quad\begin{cases}1:\text{limits}\\ 1:\ \text{integrand}\end{cases}\\ 1:\ \text{answer}\end{cases}$$

(c)

$$\text{Volume} = \int_1^8 \frac{\pi}{8}y^2\,dx = \int_1^8 \frac{\pi}{8}\left(\frac{8}{\sqrt[3]{x}}\right)^2 dx$$

$$= \int_1^8 8\pi x^{-2/3}\,dx = 8\pi\left(\frac{x^{1/3}}{1/3}\right)_1^8$$

$$= 24\pi\sqrt[3]{x}\Big|_1^8 = 24\pi(2-1) = 24\pi = 75.398$$



$$3:\begin{cases}1:\text{limits}\\ 1:\text{integrand}\\ 1:\text{answer}\end{cases}$$

2. p. 141

(a) To find the maximum velocity we first determine the critical numbers.

The derivative $v'(t) = \dfrac{(1+t^2) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2} = \dfrac{(1+t)(1-t)}{(1+t^2)^2}$ exists and is continuous everywhere. Thus, the only critical numbers are inputs for which $v'(t) = 0$. From the factorization of v' it is clear that $v'(t) = 0 \;\Rightarrow\; t = \pm 1$.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

If $t < 1$, then $v'(t) > 0$ and f is increasing; if $t > 1$, then $v'(t) < 0$ and f is decreasing. Thus, f attains a relative maximum at $t = 1$. In fact, since f always increases to the left of $t = 1$ and always decreases to the right of $t = 1$, it is clear that f attains its maximum value at $t = 1$ at which its maximum velocity is

$$v(1) = 1 + \frac{1}{1+t^2} = \frac{3}{2} = 1.5\,.$$

(b) The position is found by integrating the velocity function.

$$s(t) = \int 1 + \frac{t}{1+t^2}\,dt = t + \frac{1}{2}\ln(1+t^2) + C.$$

At time $t = 0$, the particle is at $x = 1$, thus

$1 = 0 + \frac{1}{2}\ln(1) + C$ and $C = 1$.

Hence, $s(t) = t + \frac{1}{2}\ln(1+t^2) + 1$

$3: \begin{cases} 1: \int v(t)\,dt \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(c)

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(1 + \frac{\frac{1}{t}}{\frac{1}{t^2}+1}\right) = 1 + \frac{0}{0+1} = 1$$

1 : answer

(d) The position at time t is $s(t) = t + \frac{1}{2}\ln(1+t^2) + 1$.

Solving $t + \frac{1}{2}\ln(1+t^2) + 1 = 101$ or $t + \frac{1}{2}\ln(1+t^2) + 1 - 100 = 0$ using the graphing calculator gives $t = 95.441$.

$2: \begin{cases} 1: \text{solves } s(t) = 101 \\ 1: \text{answer} \end{cases}$

3. p. 142

(a) Divide [0, 8] into four equal subintervals of [0, 2], [2, 4], [4, 6], and [6, 8] with
midpoints $m_1 = 1$, $m_2 = 3$, $m_3 = 5$, and $m_4 = 7$.

The Riemann sum is $S_4 = \sum_{i=1}^{4} \Delta t \cdot R(m_i)$
where $R(1) = 5.4$, $R(3) = 6.5$, $R(5) = 6.3$, and $R(7) = 5.5$.

$S_4 = 2(5.4) + 2(6.5) + 2(6.3) + 2(5.5) = 47.4$ metric tons pumped out.

(b) Since R(2) = R(6) = 6.1, the Mean Value Theorem guarantees a time t between t = 2 and t = 6 when $R'(t) = \frac{R(6) - R(2)}{6 - 2} = 0$.

(c)
$$Q_{\text{ave}} = \frac{1}{8-0}\int_0^8 \frac{1}{8}(36 + 8t - t^2)\, dt$$

$$= \frac{1}{64}\left(36t + 4t^2 - \frac{t^3}{3}\right]_0^8 = 5.833 \text{ metric tons per hour.}$$

1: units
metric tons in (a) and
metric/tons per hr in (c)

$3: \begin{cases} 1: R(1) + R(3) + R(5) + R(7) \\ 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

$2: \begin{cases} 1: \text{yes} \\ 1: \text{MVT or equivalent} \end{cases}$

$3: \begin{cases} 1: \text{limits and average value constant} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

Exam VI
Section II
Part B — No Calculators

4 p. 143

(a) The acceleration is the rate of change of velocity; that is, the slope of the velocity graph. The rocket's fuel was expended at $t = 4$ at which point it continued upward but slowed down until $t = 7$ when it started to fall. Thus the acceleration of the rocket during the first 4 seconds is

$a = v' = \dfrac{96-0}{4-0} = 24$ ft per sec^2

$$3: \begin{cases} 1: \text{fuel expended at } t = 4 \\ 1: v'(t) = \text{line slope} \\ 1: \text{answer / units} \end{cases}$$

(b) The rocket goes up while v is positive. $v > 0$ for $0 < t < 7$ seconds. The rocket rises for 7 seconds.

$$2: \begin{cases} 1: v(t) > 0 \\ 1: \text{answer} \end{cases}$$

(c) The distance traveled on the interval [0, 7] is

$\int_0^7 v(t)\,dt =$ the area of the triangle above the x-axis $= \frac{1}{2}(7)(96) = 336$ feet.

$$2: \begin{cases} 1: \text{triangle area} = \int_0^7 v(t)\,dt \\ 1: \text{answer} \end{cases}$$

(d) The height of the tower is the difference between the distance the rocket falls on $7 < t < 14$ and the distance it rises from part (c).

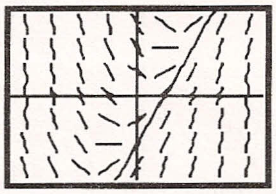
$\int_7^{14} v(t)\,dt =$ area of the triangle below the x-axis $= \frac{1}{2}(7)(224) = 784$ feet.

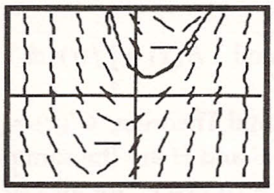
The height of the tower is $784 - 336 = 448$ feet.

$$2: \begin{cases} 1: \text{triangle area} \\ 1: \text{answer} \end{cases}$$

5 p. 144

(a) solution curve passing through (1,0). solution curve passing through (0,1)





$2: \begin{cases} 1\text{: curve through } (1,0) \\ 1\text{: curve through } (0,1) \end{cases}$

(b)

If $y = 2x + b$ is to be a solution, then $\frac{dy}{dx} = 2$. But we are given that $\frac{dy}{dx} = 2x - y$. Hence $2 = 2x - y$. Therefore $y = 2x - 2$ and $b = -2$.

1: answer

(c) At the point (0,0), $\frac{dy}{dx} = 2x - y = 0$. Therefore the graph of g has a horizontal tangent at (0,0). Differentiating $\frac{dy}{dx}$ implicitly, we have

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y)$$

At the point (0,0), this gives $\frac{d^2y}{dx^2} = 2$. That means that the graph of g is concave up at the origin. Since g has a horizontal tangent and is concave up at the origin, it has a local minimum there.

$4: \begin{cases} 1\text{: graph of g has a horizontal tangent at } (0,0) \\ 1\text{: show } g''(0) = 2 \\ 1\text{: answer} \\ 1\text{: justification} \end{cases}$

(d) The derivative of $y = Ce^{-x} + 2x - 2$ is $\frac{dy}{dx} = -Ce^{-x} + 2$. Substituting the given equation $\frac{dy}{dx} = 2x - y$ gives $2x - y = -Ce^{-x} + 2$. Solving for y we obtain $y = Ce^{-x} + 2x - 2$, which is the given solution.

$2: \begin{cases} 1\text{: } \frac{dy}{dx} \\ 1\text{: substitution} \end{cases}$

6 p. 145

(a)

$$G(x) = \int_{-3}^{x} f(t)\ dt \quad \text{and} \quad H(x) = \int_{2}^{x} f(t)\ dt$$

By the Second Fundamental Theorem, $G'(x) = f(x)$ and $H'(x) = f(x)$. Since the derivatives of G and H are the same, G and H differ by a constant. That is, $G(x) - H(x) = C$ for all x in the domain. This means that the graph of G is the same as the graph of H, moved C units vertically.

To evaluate the constant C, note that: $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$.

In particular, $\int_{-3}^{x} f(t)\,dt = \int_{-3}^{2} f(t)\,dt + \int_{2}^{x} f(t)\,dt$.

Thus $G(x) = \int_{-3}^{2} f(t)\,dt + H(x)$. Hence $G(x) - H(x) = \int_{-3}^{2} f(t)\,dt$. Using the areas of the triangles $C = \int_{-3}^{2} f(t)\,dt = -\frac{1}{2}(2)(3) + \frac{1}{2}(1)(1) = -\frac{7}{2}$.

$2: \begin{cases} 1: \text{G(x) and H(x) differ by a constant} \\ 1: \text{geometric explanation} \end{cases}$

(b)

H is increasing $\Leftrightarrow$ $H'(x) = f(x) > 0$.
This occurs on the intervals (-5,-3) and (1,5).

$3: \begin{cases} 1: H'(x) = f(x) > 0 \\ 2: \text{answer} \end{cases}$

(c) G has relative maximum values wherever H does. Because of the result of part (b), this will be at $x = -3$, where g'(x) changes sign from plus to minus.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) G is concave up $\Leftrightarrow$ $G'(x) = f(x)$ is increasing.
This occurs on the intervals (-2, 1) and (1, 3).
G"(1) doesn't exist because $f'(1^-) = \frac{2}{3}$ while $f'(1^+) = 1$.

$2: \begin{cases} 1: G'(t) = f(t) \text{ is increasing} \\ 1: \text{intervals} \end{cases}$